Metric dimension of dual polar graphs

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Abstract

A *resolving set* for a graph Γ is a collection of vertices S, chosen so that for each vertex v, the list of distances from v to the members of S uniquely specifies v. The *metric dimension* $\mu(\Gamma)$ is the smallest size of a resolving set for Γ. We consider the metric dimension of the *dual polar graphs*, and show that it is at most the rank over $\mathbb{R}$ of the incidence matrix of the corresponding polar space. We then compute this rank to give an explicit upper bound on the metric dimension of dual polar graphs.

1 Introduction

1.1 Resolving sets and metric dimension

Let Γ denote a graph with vertex set V and edge set E, which we assume to be finite, connected, loopless, and with no multiple edges. A *resolving set* for Γ is a subset $S \subseteq V$ with the property that, for every $u \in V$, the list of distances from u to each of the elements of S uniquely identifies u; equivalently, for two distinct vertices $u, w \in V$, there exists $x \in S$ for which $\mathrm{d}(u,x) \neq \mathrm{d}(w,x)$. (Here, $d(x,y)$ denotes the length of a shortest path from x to y in Γ.) The *metric dimension* of Γ is the smallest size of a resolving set for Γ, and we denote this by $\mu(\Gamma)$. These notions were introduced to graph theory in the 1970s by Slater [22] and, independently, by Harary and Melter [18]; in more general metric spaces, the concept can be found in the literature much earlier (see [8]).

The 2011 survey article by Cameron and the first author [5] was the first to consider, systematically, the metric dimension of distance-regular graphs. Since then various papers have been written on this subject, as well as the more general problem of class dimension of association schemes, often focused on particular families: for example, Johnson and Kneser graphs [4], Grassmann graphs [6], bilinear forms graphs [11] and symplectic dual polar graphs [15]; see also [1, 2, 3, 7, 12, 13, 14, 16, 17, 19]. In this paper, we will consider *dual polar graphs*, which are defined below.

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1.2 Polar spaces and dual polar graphs

Let $V(n, q)$ denote the vector space of dimension n over $\mathbb{F}_q$, the finite field with q elements, equipped with either a symplectic, quadratic or Hermitian non-degenerate structure. Recall that to define one of these three structures on $V(n, q)$, we require a sesquilinear form $b : V(n, q) \times V(n, q) \to \mathbb{F}_q$ in two variables and/or a quadratic form $f : V(n, q) \to \mathbb{F}_q$ in one variable, where f is either defined by $f(v) = b(v, v)$ for every $v \in V(n, q)$, or b is obtained by polarising f. A subspace U of $V(n, q)$ is called *totally isotropic* if the restriction to U of either the sequilinear form or the quadratic form is identically zero; that is, $b(v, w) = 0$ or $f(v) = 0$ for all $v, w \in U$. A *(classical) polar space* $\mathcal{P}$ is the collection of all totally isotropic subspaces of $V(n, q)$. The *Witt index* of $V(n, q)$ is the dimension of a largest totally isotropic subspace of $V(n, q)$. The Witt index of $V(n, q)$ is often called the *rank* of $\mathcal{P}$. The 1-dimensional totally isotropic subspaces are the *points* of $\mathcal{P}$, while if $\mathcal{P}$ has Witt index d, the d-dimensional totally isotropic subspaces are the *generators* (or the *maximals*) of $\mathcal{P}$.

With this terminology, we make the following definition.

Definition 1. Let $\mathcal{P}$ be a polar space over $\mathbb{F}_q$ with Witt index d. The *dual polar graph* on $\mathcal{P}$ has the generators of $\mathcal{P}$ as vertices, and two generators are adjacent if and only if their intersection has dimension $d - 1$.

There are six families of classical polar spaces, and thus six families of dual polar graphs, arising from the classification of sequilinear and quadradic forms. Numerical information about these spaces can be expressed in terms of the field order q, Witt index d, and a parameter $e \in \{0,\ 1/2,\ 1,\ 3/2,\ 2\}$ depending on the choice of form. We note that a Hermitian polar space requires the field order q to be a square. These are summarized in the table below; our notation follows [9].

Polar space	Names	Vector space	e
$[C_d(q)] \cong Sp(2d, q)$	Symplectic	$V(2d, q)$	1
$[B_d(q)] \cong \Omega(2d + 1, q)$	Orthogonal; parabolic quadric	$V(2d + 1, q)$	1
$[D_d(q)] \cong \Omega^+(2d, q)$	Orthogonal; hyperbolic quadric	$V(2d, q)$	0
$[^2D_{d+1}(q)] \cong \Omega^-(2d + 2, q)$	Orthogonal; elliptic quadric	$V(2d + 2, q)$	2
$[^2A_{2d}(\sqrt{q})] \cong U(2d + 1, q)$	Unitary; Hermitian variety	$V(2d + 1, q)$	3/2
$[^2A_{2d-1}(\sqrt{q})] \cong U(2d, q)$	Unitary; Hermitian variety	$V(2d, q)$	1/2

Table 1: Classical polar spaces

We refer the reader to Cameron [10] for background on polar spaces, and to Brouwer, Cohen and Neumaier [9, Section 9.4] for dual polar graphs. We will use the notation $\Gamma(q, d, e)$ to denote a dual polar graph when the type is unspecified. The following results are all taken from [9, Section 9.4].

Lemma 2. *Let $\Gamma(q, d, e)$ be the dual polar graph arising from a polar space $\mathcal{P}$. Then:*

(a) The number of points of $\mathcal{P}$ is

$$\frac{(q^{d+e-1} + 1)(q^d - 1)}{q - 1}.$$

(b) The number of generators of $\mathcal{P}$, and thus the number of vertices of $\Gamma(q,d,e)$, is

$$\prod_{i=0}^{d-1}(q^{e+i}+1).$$

(c) If U, W are vertices of $\Gamma(q,d,e)$, then U, W are at distance i if and only if $\dim(U \cap W) = d-i$.

We note that when $e=1$, the dual polar graphs $[C_d(q)]$ and $[B_d(q)]$ have the same parameters, but are not isomorphic in general.

2 The main theorem

In [4, 6], the first author, Meagher and others considered the metric dimension of Johnson graphs and Grassmann graphs, and obtained upper bounds on this equal to the rank of an appropriate incidence matrix. This approach was subsequently used in [12, 13, 14] for the class dimension of various families of association schemes. In what follows, we shall adapt this technique for dual polar graphs.

Suppose that we have a dual polar graph $\Gamma(q,d,e)$ arising from a polar space $\mathcal{P}$. For $t \in \{1,\ldots,d\}$, let Ω_t denote the set of all totally isotropic t-dimensional subspaces in $\mathcal{P}$ (so that Ω_1 is the set of points of $\mathcal{P}$, and Ω_d the set of generators of $\mathcal{P}$).

We recall (from [9, Section 1.3], for instance) that a graph is *strongly regular* with parameters (n,k,a,c) if it has n vertices, is regular with degree k, any pair of adjacent vertices have a common neighbours, and any pair of non-adjacent vertices have c common neighbours. Polar spaces are a source of such graphs: the *collinearity graph* of a polar space $\mathcal{P}$ is the graph whose vertices are the points of $\mathcal{P}$, and where two distinct points are adjacent if and only if their span is totally isotropic. The following facts will be of use to us.

Lemma 3. *Let Δ denote the collinearity graph of a polar space $\mathcal{P}$ with parameters q,d,e as above. Then Δ is strongly regular with parameters (n,k,a,c), where*

$$\begin{aligned}
n &= |\Omega_1| = (q^{d+e-1}+1)\frac{q^d-1}{q-1},\\
k &= q(q^{d+e-2}+1)\frac{q^{d-1}-1}{q-1},\\
a &= (q-1)+q^2(q^{d+e-3}+1)\frac{q^{d-2}-1}{q-1},\\
c &= (q^{d+e-2}+1)\frac{q^{d-1}-1}{q-1}.
\end{aligned}$$

Furthermore Δ has eigenvalues $\theta_0=k$, $\theta_1=q^{d-1}+1$ and $\theta_2=-(q^{d+e-2}+1)$, with multiplicities $m_0=1$, m_1 and m_2 (respectively), where

$$\begin{aligned}
m_1 &= \frac{q^e(q^{d+e-2}+1)}{q^{e-1}+1}\,\frac{q^d-1}{q-1},\\
m_2 &= \frac{q(q^{d+e-1}+1)}{q^{e-1}+1}\,\frac{q^{d-1}-1}{q-1}.
\end{aligned}$$

Proof. The fact that Δ is strongly regular with the parameters (n,k,a,c) as given is well-known (see, for example, [20, 21]). To calculate the eigenvalues and their multiplicities, we use the standard formulas for strongly regular graphs (see [9, Theorem 1.3.1]); in particular, θ_1 and θ_2 are the roots of the polynomial $T^2+(c-a)T+(c-k)=0$, and for the multiplicities we use the formulas

$$m_1 = \frac{(\theta_2+1)k(k-\theta_2)}{c(\theta_2-\theta_1)}, \qquad m_2 = n - m_1 - 1.$$

□

For a given $U \in \Omega_t$, the *incidence vector* of U is the vector $\mathbf{u} \in \mathbb{R}^{\Omega_1}$ with entries 0 or 1 so that, for any $x \in \Omega_1$, the x-coordinate of $\mathbf{u}$ is 1 if $x \subseteq U$ and 0 otherwise. If we have a collection of totally isotropic subspaces $\mathcal{W} = \{W_1,\ldots,W_m\}$, the *incidence matrix* of $\mathcal{W}$ is the $m \times |\Omega_1|$ matrix whose rows are the incidence vectors of $W_1,\ldots,W_m$. We shall refer to the incidence matrix of the collection of all generators of $\mathcal{P}$ as the *incidence matrix of* $\mathcal{P}$. The following lemma will be crucial in our upper bound on the size of a resolving set for $\Gamma(q,d,e)$.

Lemma 4. *Let $\mathcal{P}$ be a polar space with parameters q,d,e as above. Then the incidence matrix of $\mathcal{P}$ has rank*

$$\frac{(q^{d+e-1}+1)(q^{d+e-1}-q^{e-1}+q-1)}{(q^{e-1}+1)(q-1)}.$$

Proof. Let M be the incidence matrix of $\mathcal{P}$ and let $B = M^TM$. By standard linear algebra, $\mathrm{rank}(B) = \mathrm{rank}(M)$. Clearly, B is an $|\Omega_1| \times |\Omega_1|$ matrix with rows and columns indexed by the points of $\mathcal{P}$. For each $t \in \{0,\ldots,d-1\}$, let N_t denote the number of generators containing a fixed t-dimensional subspace U. Observe that $U^\perp/U$ is a polar space of the same type as $\mathcal{P}$ with parameters $q, d-t, e$. Hence, using Lemma 2, we have that $N_t = \prod_{i=0}^{d-t-1}(q^{e+i}+1)$. Moreover, for two points $x,y \in \Omega_1$, the entry B_{xy} is the number of generators containing both x and y, so there are three possible values for this:

$$B_{x,y} = \begin{cases} N_1 & \text{if } x = y, \\ N_2 & \text{if } x \neq y \text{ and the span of } x \text{ and } y \text{ is totally isotropic,} \\ 0 & \text{if } x \neq y \text{ and the span of } x \text{ and } y \text{ is not totally isotropic.} \end{cases}$$

From this, it follows that

$$B = N_1 I + N_2 A,$$

where A is the adjacency matrix of the collinearity graph of $\mathcal{P}$. Therefore, the eigenvalues of B are $\lambda_i = N_1 + \theta_i N_2$, where θ_i is an eigenvalue of A, and λ_i and θ_i have the same multiplicity. The eigenvalues $\theta_0, \theta_1, \theta_2$ of A and their multiplicities were given in Lemma 3, from which we observe that $N_1 + \theta_2 N_2 = 0$. Consequently, B is a singular matrix, and its nullity is equal to m_2 (the multiplicity of θ_2). Therefore, we have

$$\mathrm{rank}(M) = \mathrm{rank}(B) = |\Omega_1| - m_2 = \frac{(q^{d+e-1}+1)(q^{d+e-1}-q^{e-1}+q-1)}{(q^{e-1}+1)(q-1)}.$$

□

By Lemma 2(c), a collection $\mathcal{S} = \{X_1, \ldots, X_m\}$ of generators of $\mathcal{P}$ will form a resolving set for $\Gamma(q, d, e)$ if and only if the map $\Omega_d \to \mathbb{R}^m$ defined by $U \mapsto (\dim(X_1 \cap U), \ldots, \dim(X_m \cap U))$ is injective. Now, for any totally isotropic subspaces U, W, we have that $\dim(U \cap W) = k$ if and only if U and W have $(q^k - 1)/(q - 1)$ points of $\mathcal{P}$ in common. Thus we can phrase the "resolving property" for a collection of generators in terms of linear algebra: if M is the incidence matrix of a collection of generators $\mathcal{S} = \{X_1, \ldots, X_m\}$, and $\mathbf{u}$ is the incidence vector (as a column vector) of a given generator U, then the entry in position i of the vector $M\mathbf{u} \in \mathbb{R}^m$ is $(q^{\dim(X_i \cap U)} - 1)/(q - 1)$. Consequently, $\mathcal{S} = \{X_1, \ldots, X_m\}$ is a resolving set with incidence matrix M if and only if, for any generators U, W with incidence vectors $\mathbf{u}, \mathbf{w}$, we have that $M\mathbf{u} = M\mathbf{w}$ implies that $U = W$.

Theorem 5. *Let $\Gamma(q, d, e)$ be the dual polar graph arising from a polar space $\mathcal{P}$. Then the metric dimension of $\Gamma(q, d, e)$ is at most the rank, over $\mathbb{R}$, of the incidence matrix of $\mathcal{P}$, that is, it is at most*

$$\frac{(q^{d+e-1} + 1)(q^{d+e-1} - q^{e-1} + q - 1)}{(q^{e-1} + 1)(q - 1)}.$$

Proof. Let M denote the incidence matrix of $\mathcal{P}$. Since M is an $|\Omega_d| \times |\Omega_1|$ matrix, it has more rows than columns, and thus $\operatorname{rank}(M) \leq |\Omega_1|$. By rearranging rows if necessary, we can assume that

$$M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

where M_1 is a $\operatorname{rank}(M) \times |\Omega_1|$ matrix whose rows are linearly independent. We will show that M_1 is the incidence matrix of a resolving set for $\Gamma(q, d, e)$.

By construction, we have $\operatorname{rank}(M_1) = \operatorname{rank}(M)$; since both matrices have the same number of columns, it follows that they have the same nullity. However, if $M\mathbf{x} = \mathbf{0}$ we must have $M_1\mathbf{x} = \mathbf{0}$, and thus $\ker(M) \subseteq \ker(M_1)$; as $\ker(M)$ and $\ker(M_1)$ have the same dimension, it follows that $\ker(M) = \ker(M_1)$. Now suppose that U, W are generators of $\mathcal{P}$ with incidence vectors $\mathbf{u}, \mathbf{w}$, respectively. Then we have

$$\begin{aligned} M_1\mathbf{u} = M_1\mathbf{w} \quad &\iff \quad M_1(\mathbf{u} - \mathbf{w}) = \mathbf{0} \\ &\iff \quad \mathbf{u} - \mathbf{w} \in \ker(M_1) \\ &\iff \quad \mathbf{u} - \mathbf{w} \in \ker(M) \\ &\iff \quad M\mathbf{u} = M\mathbf{w}. \end{aligned}$$

Therefore we must have $\dim(U \cap Z) = \dim(W \cap Z)$ for all $Z \in \Omega_d$. In particular, this holds for $Z = U$, so $\dim(W \cap U) = \dim(U)$ and thus $U = W$.

Hence M_1 is indeed the incidence matrix of a resolving set for $\Gamma(q, d, e)$, of size $\operatorname{rank}(M)$. The rest of the proof follows immediately from Lemma 4. □

In Table 2 we restate the bound on the metric dimension according on the type of the polar space.

3 Final remarks

The only case of dual polar graphs for which there is an existing bound in the literature is the symplectic dual polar graph on $[C_q(q)]$, which was considered by Guo, Wang and Li [15] in 2013;

Graph	Bound on metric dimension
$[C_d(q)]$, $[B_d(q)]$	$\frac{1}{2}\frac{(q^d+1)(q^d+q-2)}{q-1}$
$[D_d(q)]$	$\frac{(q^{d-1}+1)(q^d+q^2-q-1)}{q^2-1}$
$[^2D_{d+1}(q)]$	$\frac{q^{2(d+1)}-1}{q^2-1}$
$[^2A_{2d}(\sqrt{q})]$	$\frac{(q^{d+\frac{1}{2}}+1)(q^{d+\frac{1}{2}}+q-\sqrt{q}-1)}{(\sqrt{q}+1)(q-1)}$
$[^2A_{2d-1}(\sqrt{q})]$	$\frac{(q^{d-\frac{1}{2}}+1)(q^d+q^{\frac{3}{2}}-\sqrt{q}-1)}{(\sqrt{q}+1)(q-1)}$

Table 2: Bounds for each family of dual polar graphs

in that paper, the authors gave an upper bound on its metric dimension of $(q^d+1)(q^d+q-2)/(q-1)$, which is exactly double our bound.

The dual polar graphs $[D_d(q)]$ are bipartite; as such, one may consider their *halved graphs*, which are the connected components of the distance-2 graphs. It would be interesting to see if the techniques used above could be used to determine the metric dimension of the halved graphs, as well as other related graphs such as *Ustimenko graphs* and *Hemmeter graphs*. (See [9, Section 9.4C] for further details on these graphs.)

If examined more closely, Theorem 5 is actually a result concerning the class dimension of association schemes arising from polar spaces. (See [5, Section 3.4] for more details on class dimension.) Indeed, if minded so, given a polar space with parameters q, d, e and some $t \in \{1, \dots, d\}$, one might consider the graph having the collection of the totally isotropic subspaces of dimension t as vertices, and where two such subspaces are adjacent if their intersection has codimension 1. In particular, the dual polar graphs correspond to the case $t = d$. With minor modifications, the argument in the proof of Theorem 5 yields a bound on the $|\Omega_1| \times |\Omega_t|$ incidence matrix of the corresponding incidence structure. However, we have preferred to phrase our results only for the case $t = d$, i.e. for dual polar graphs, because only in this case the graph is distance-regular and hence only in this case our bound on the rank yields a bound on the metric dimension.

As far as we are aware, the problem of bounding from below the metric dimension of dual polar graphs is entirely open. Here we dare to conjecture that, there exists a positive constant c (which does not depend on the dual polar graph $\Gamma(q, d, e)$), such that the metric dimension is at least $c \cdot M(q, d, e)$, where $M(q, d, e)$ is the upper bound in Theorem 5.

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