

Hamiltonian decompositions of complete k -uniform hypergraphs

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Abstract

Using a generalisation of Hamiltonian cycles to uniform hypergraphs due to Katona and Kierstead, we define a new notion of a Hamiltonian decomposition of a uniform hypergraph. We then consider the problem of constructing such decompositions for complete uniform hypergraphs, and describe its relationship with other topics, such as design theory.

Keywords: Uniform hypergraph; Hamiltonian cycle; Hamiltonian chain; Hamiltonian decomposition; large set; directed terrace; difference pattern.

MSC2000: 05C65 (primary), 05B05 (secondary).

1 Introduction

A *decomposition* of a graph $G = (V, E)$ is a partition of the edge-set E ; a *Hamiltonian decomposition* of G is a decomposition into Hamiltonian cycles. The problem

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of constructing Hamiltonian decompositions is a long-standing and well-studied one in graph theory; in particular, for the complete graph K_n , it was solved in the 1890s by Walecki. (See Lucas [18] or the recent articles by Alspach [1] and Bryant [7] for details.) Walecki showed that K_n has a Hamiltonian decomposition if and only if n is odd, while if n is even K_n has a decomposition into Hamiltonian cycles and a perfect matching.

As with many problems in graph theory, it seems natural to attempt a generalisation to hypergraphs. Indeed, the notion of Hamiltonicity was first generalised to uniform hypergraphs by Berge in his 1970 book [5]. His definition of a Hamiltonian cycle in a hypergraph $\mathcal{H} = (V, E)$ is a sequence $(v_0, e_1, v_1, e_2, \dots, v_{n-1}, e_n, v_0)$, where $\{v_0, \dots, v_{n-1}\} = V$, and e_1, \dots, e_n are distinct elements of E , such that the hyperedge e_i contains both v_{i-1} and v_i (modulo n). The study of decompositions of complete 3-uniform hypergraphs into cycles of this type was begun by Bermond *et al* in the 1970s [6] and was completed by Verrall in 1994 [22]. We will consider a different notion of Hamiltonicity which will be defined in the next section, although there are others besides these, such as the *loose Hamilton cycles* defined by Kühn and Osthus [17].

2 Definitions

We begin by defining the objects we'll be discussing throughout the paper. A *hypergraph* $\mathcal{H} = (V, \mathcal{E})$ consists of a finite set V of *vertices* with a family \mathcal{E} of subsets of V , called *hyperedges* (or simply *edges*). If each (hyper)edge has size k , we say that \mathcal{H} is a *k-uniform* hypergraph. In particular, the *complete k-uniform hypergraph* on n vertices has all k -subsets of $\{1, \dots, n\}$ as edges; we denote this by $K_n^{(k)}$.

The notion of Hamiltonicity was generalised to k -uniform hypergraphs by G. Y. Katona and Kierstead in their 1999 paper [15] as follows:

Definition 1. Let $\mathcal{H} = (V, \mathcal{E})$ be a k -uniform hypergraph. A *Hamiltonian cycle* in \mathcal{H} is a cyclic ordering of the elements of V such that each consecutive k -tuple of vertices is an edge.

(In fact, Katona and Kierstead use the term *Hamiltonian chain* instead of Hamiltonian cycle.) We observe that one can define a cycle in general by considering an arbitrary closed sequence of vertices. If one wishes to generalise the notion of an Eulerian cycle to hypergraphs in this way, one obtains precisely a *universal cycle*; these have been studied for $K_n^{(k)}$ by Chung, Diaconis and Graham [10] and by Hurlbert [13], for example.

Returning to Hamiltonicity, the following definition is an obvious generalisation of one for graphs.

Definition 2. A *Hamiltonian decomposition* of a hypergraph \mathcal{H} is a partition of the set of (hyper)edges of \mathcal{H} into mutually-disjoint Hamiltonian cycles.

We are concerned with finding Hamiltonian decompositions of the complete k -uniform hypergraph $K_n^{(k)}$. We need only consider the case $k \geq 3$, as $k = 1$ is a degenerate case, while $k = 2$ (i.e. the complete graph) was solved by Walecki. Also, we observe that by taking the complements of the edges in a Hamiltonian cycle in $K_n^{(k)}$ we obtain a Hamiltonian cycle in $K_n^{(n-k)}$; hence if we have a Hamiltonian decomposition of $K_n^{(k)}$ we also have one of $K_n^{(n-k)}$. Thus it suffices to consider $3 \leq k \leq \frac{n}{2}$. There are also obvious necessary numerical conditions, as we see below.

Lemma 3. *If a Hamiltonian decomposition of $K_n^{(k)}$ exists, then $n \mid \binom{n}{k}$.*

Proof. Clearly, the number of edges in a Hamiltonian cycle (which is n) must divide the total number of edges of $K_n^{(k)}$ (which is $\binom{n}{k}$). Hence $n \mid \binom{n}{k}$. Also, the number of times a vertex v appears in an edge of a Hamiltonian cycle (which is k) must divide the total number of edges containing v (which is $\binom{n-1}{k-1}$). Hence $k \mid \binom{n-1}{k-1}$; however, this can easily be seen to be equivalent to $n \mid \binom{n}{k}$. \square

We call the parameters (n, k) *feasible* if the above condition is satisfied. Clearly, $(n, 2)$ (and hence $(n, n-2)$) are feasible if and only if n is odd, while $(n, 3)$ (and hence $(n, n-3)$) are feasible if and only if n is not a multiple of 3.

Based on the evidence which follows in the remainder of this paper, we make the following conjecture.

Conjecture. *Let $n \geq 5$ and $2 \leq k \leq n-2$. Then there exists a Hamiltonian decomposition of $K_n^{(k)}$ if and only if $n \mid \binom{n}{k}$.*

In other words, we conjecture that the obvious necessary condition is also sufficient.

We conclude this section by remarking on a related conjecture of Baranyai and G. O. H. Katona (see [14], Conjecture 4.1). Suppose k does not divide n . A *wreath* in $K_n^{(k)}$ is a sequence of edges isomorphic to

$$\{1, \dots, k\}, \{k+1, \dots, 2k\}, \dots, \{(a-1)k, \dots, ak\}$$

(where $a = \text{lcm}(n, k)/k$, and addition is modulo n). If n and k are coprime, then this is exactly a Hamiltonian cycle. Baranyai and Katona conjectured that $K_n^{(k)}$ can be partitioned into disjoint wreaths, so when $\text{gcd}(n, k) = 1$ their conjecture is equivalent to ours. However, when n and k are not coprime, the two conjectures are quite different.

3 Clique-finding

When presented with a question such as this, one might ask how to construct the desired object by computer. A commonly-used technique is to devise a graph, and to search the graph for a maximum clique (see the survey by Östergård [20] for some examples). Our initial experiments in searching for Hamiltonian decompositions of $K_n^{(k)}$ utilised this approach.

We construct a graph $\Gamma_{n,k}$ as follows. The vertex set of $\Gamma_{n,k}$ will be the set of all possible Hamiltonian cycles of $K_n^{(k)}$. Hence the graph has $\frac{1}{2}(n-1)!$ vertices, as there are $(n-1)!$ cyclic orderings of n objects, and reversing the ordering of a Hamiltonian cycle gives the same set of (hyper)edges. Then we join two vertices in $\Gamma_{n,k}$ if and only if the corresponding cycles are disjoint (i.e. they have no (hyper)edge in common). Thus a clique in $\Gamma_{n,k}$ corresponds to a set of mutually disjoint Hamiltonian cycles in $K_n^{(k)}$. Furthermore, if there exists a clique of size $\binom{n}{k}/n$, then this corresponds to a Hamiltonian decomposition of $K_n^{(k)}$ (consequently, this is the maximum possible size of a clique in $\Gamma_{n,k}$).

Using the GRAPE package [21] for the GAP computer algebra system [11], it is straightforward to construct the graph $\Gamma_{n,k}$. Also, GRAPE has an in-built command, `CliquesOfGivenSize`, to find cliques of a specified size in a given graph. Using these commands, we were able to feed the feasible parameter sets $(7,3)$, $(8,3)$ and $(9,4)$ to a computer (these being the smallest cases not already handled by Walecki's result), and show that the complete hypergraphs $K_7^{(3)}$, $K_8^{(3)}$ and $K_9^{(4)}$ each admit Hamiltonian decompositions.

Example 4. The following $\binom{7}{3}/7 = 5$ orderings, viewed cyclically, give a Hamiltonian decomposition of $K_7^{(3)}$:

1	2	4	6	7	5	3
1	2	6	3	4	7	5
1	3	4	5	6	2	7
1	4	5	2	7	3	6
1	6	5	3	2	4	7

Examples of Hamiltonian decompositions of $K_8^{(3)}$ and $K_9^{(4)}$ are given in Appendix A.

Unfortunately for us, when running on a typical 2004-model desktop PC, this approach runs out of steam after this stage: a combinatorial explosion means that the number of vertices of $\Gamma_{n,k}$ and the size of the clique being sought are too large

for the computer to handle (in terms of both memory and time), so we had to switch to a high-performance computing facility. At the time of writing, a search for a clique in $\Gamma_{10,4}$ is in progress. However, a different approach is really what is needed.

4 Design theory: large sets

It is possible to rephrase the idea of a Hamiltonian cycle in a hypergraph in the language of block designs. A t - (v, k, λ) design (V, \mathcal{B}) consists of a set V of v points together with a family \mathcal{B} of k -subsets of V , called *blocks*, with the property that any t -subset of points is contained in exactly λ blocks. So therefore any t -design is also a k -uniform hypergraph, where points are vertices and blocks are (hyper)edges.

A *large set* of t -designs is a partition of the complete k -uniform hypergraph on v vertices (often called the *complete design* in this context) into t - (v, k, λ) designs; see the survey by Khosrovshahi and Tayfeh-Rezaie [16] for full details.

We notice a Hamiltonian cycle in $K_n^{(k)}$ is an example of a 1 - (n, k, k) design; clearly, each vertex (i.e. point) lies in exactly k edges. Therefore a Hamiltonian decomposition of $K_n^{(k)}$ is, in the language of design theory, a large set of 1 - (n, k, k) designs. So one may ask what known results in the design theory literature may be of use to us here.

In 1987, Hartman [12] showed that large sets of 1 - (v, k, λ) designs exist if and only if the obvious necessary numerical conditions (that is, a more general version of our Lemma 3) are satisfied. Hartman proves this as a corollary to *Baranyai's Partition Theorem* [4] (see also Cameron [8]).

Unfortunately, as Baranyai's theorem has a non-constructive existence proof, Hartman's result doesn't give any hint as to the structure of the 1 -designs obtained, or even if the 1 -designs in a large set can be assumed to be isomorphic. Consequently, it seems unlikely that Hartman's approach can be modified in order to demonstrate the existence of Hamiltonian decompositions. However, it does give some credence to our conjecture: if Hartman's theorem showed that there were feasible parameter sets that did not admit large sets of 1 - (v, k, k) designs, then those parameters would therefore not admit Hamiltonian decompositions (and thus provide counterexamples to our conjecture).

One point of interest concerns large sets of 1 - $(7, 3, 3)$ designs. The Fano plane is an example of a 1 - $(7, 3, 3)$ design, and it was shown by Cayley in 1850 that a large set of Fano planes does not exist [9]. However, as shown in the previous section, there *does* exist a large set of 1 - $(7, 3, 3)$ designs which are Hamiltonian cycles.

5 Difference patterns

If we regard the n vertices of $K_n^{(k)}$ as the integers modulo n , the extra structure may be of use to us. Consider the following definition:

Definition 5. A *directed terrace* for \mathbb{Z}_n is an ordering of the elements so that the set of differences between consecutive elements contains all the non-zero elements exactly once.

Directed terraces, and the more general notion of *terraces*, were defined by R. A. Bailey in 1984 [2], for arbitrary finite groups (not just \mathbb{Z}_n). The following is an example of a directed terrace for \mathbb{Z}_{12} :

$$0 \quad 11 \quad 1 \quad 10 \quad 2 \quad 9 \quad 3 \quad 8 \quad 4 \quad 7 \quad 5 \quad 6$$

We can verify that it is indeed a directed terrace by looking at the list of differences:

$$11 \quad 2 \quad 9 \quad 4 \quad 7 \quad 6 \quad 5 \quad 8 \quad 3 \quad 10 \quad 1$$

As is explained by Bailey, Ollis and Preece [3], Walecki's construction of a Hamiltonian decomposition of the complete graph K_n is equivalent to a type of directed terrace for \mathbb{Z}_{n-1} . Roughly speaking, the translation to graphs is done by labelling vertices with elements of \mathbb{Z}_{n-1} (with the remaining vertex labelled ∞), and labelling edges by differences. As this is a useful way of constructing Hamiltonian decompositions of graphs, we may ask if there is a similar method for hypergraphs.

For the rest of this section, we consider only 3-uniform hypergraphs. Now that a (hyper)edge is a triple rather than a pair of vertices, there is more than one difference to consider. Thus we make the following definition.

Definition 6. Let $T = \{a, b, c\}$ be a triple of distinct elements of \mathbb{Z}_n . Then its *difference pattern*, $\pi(T)$, is the equivalence class of ordered triples containing cyclic rotations of $(b - a, c - b, a - c)$ and $(c - a, b - c, a - b)$ (where the differences are taken modulo n).

The reason for including both of the cyclic rotations in $\pi(T)$ is that the order of the elements of T does not matter. However in our Hamiltonian cycles, each triple will appear with a fixed order. So we use the difference pattern $\pi(T)$ as a formal means to move between the ordered and unordered objects.

Clearly, the three differences sum to zero, so therefore if we know that the first two differences are x and y , then the third is $n - x - y$. By a slight abuse of notation we use $(x, y, n - x - y)$ to denote the whole equivalence class that contains it.

We can also think of difference patterns in terms of orbits. Let g be the cyclic permutation $g : x \mapsto x + 1 \pmod{n}$ of \mathbb{Z}_n . The group $\langle g \rangle$ acts on the set of all triples

in \mathbb{Z}_n , and the orbit containing a given triple T contains precisely all those triples with the difference pattern $\pi(T)$. Thus each difference pattern corresponds to an orbit of $\langle g \rangle$ on triples.

Let C be a Hamiltonian cycle in $K_n^{(3)}$. Then each edge of C has a difference pattern, and we call the list of all of these the *difference type* of C , denoted $\tau(C)$. Note that each *translate* of C , that is the cycle $C + i$ obtained by adding i (modulo n) to each vertex, has the same difference type as C (i.e. $\tau(C + i) = \tau(C)$ for all i).

Having defined difference patterns, it is natural to ask how many of them there are. This is answered below.

Lemma 7. *Suppose n is not a multiple of 3. Then the number of distinct difference patterns of triples of elements of \mathbb{Z}_n is $\binom{n}{3}/n$.*

Proof. First, there are $\binom{n}{3}$ triples of elements of \mathbb{Z}_n . Suppose $(x, y, n - x - y)$ is the difference pattern of a 3-subset T (i.e. $\pi(T) = (x, y, n - x - y)$). We observe that any other triple T' with $\pi(T') = \pi(T)$ must therefore be of the form $T + i$ for some $i \in \mathbb{Z}_n$; since n is not a multiple of 3, these are all distinct. Hence there are exactly n triples with that difference pattern, and so there must be $\binom{n}{3}/n$ distinct difference patterns altogether. \square

As a consequence, we notice that the number of cycles in a Hamiltonian decomposition of $K_n^{(3)}$ is equal to the number of distinct difference patterns of triples of elements of \mathbb{Z}_n . So we aim to construct some kind of bijection between the two. To this end, we make the following definition.

Definition 8. We call a cyclic ordering of \mathbb{Z}_n *multifarious* if its n difference patterns are all distinct.

Example 9. The following is an example of a multifarious ordering of \mathbb{Z}_{10} :

$$0 \ 1 \ 8 \ 4 \ 2 \ 3 \ 6 \ 7 \ 9 \ 5$$

By checking the 10 difference patterns (e.g. $(1, 7, 2)$, $(7, 6, 7)$, $(6, 8, 6)$, etc.), we see that they are all distinct.

This definition is a kind of generalisation of directed terraces to consider triples, rather than pairs, of adjacent elements. However, we recall that directed terraces are defined for linear orderings, rather than cyclic orderings. Also, we remark that multifarious orderings can only exist for $n \geq 10$, as we require $\binom{n}{3}/n \geq n$.

Recall that the group generated by the permutation $g : x \mapsto x + 1$ acts on the set of all triples. For any triple T , the proof of Lemma 7 shows that the orbit of $\langle g \rangle$ containing T has size n . However, $\langle g \rangle$ also acts on the set of all cyclic orderings. For a given cyclic ordering C , the orbit of $\langle g \rangle$ containing C is the set of translates

of C , $\{C + i \mid i \in \mathbb{Z}_n\}$. Now suppose C is a multifarious ordering; this forces the orbit of $\langle g \rangle$ (on orderings) containing C to have size n . Thus the set of translates $\{C + i \mid i \in \mathbb{Z}_n\}$ contains n^2 distinct triples.

At the opposite end of the scale, we also have the following.

Definition 10. We call a cyclic ordering of \mathbb{Z}_n *unary* if its difference type contains only one difference pattern, and *binary* if it contains exactly two difference patterns.

The single difference pattern found in a unary ordering is necessarily of the form $(x, x, n - 2x)$. To see this, consider a unary ordering $(0, x, y, z, w, \dots)$. For the first two difference patterns to be the same, we require that either $z = 0$, which is a contradiction, or $z = x + y$. Similarly, for the next difference triple to be the same we obtain either that $w = x = 2y$, again a contradiction, or that $y = 2x$ and $w = 4x$. A similar argument shows that if C is a binary cyclic ordering then the two difference patterns it contains must be of the form $(x, y, n - x - y)$ and $(y, x, n - x - y)$.

At this point, we introduce some more terminology.

- We call the single difference pattern $(x, x, n - 2x)$ of a unary ordering an *isosceles* difference pattern.
- We call the two difference patterns $(x, y, n - x - y)$ and $(y, x, n - x - y)$ of a binary ordering a *conjugate pair*.

Once again, we can think of these in terms of orbits of $\langle g \rangle$ on cyclic orderings. A unary cyclic ordering is in an orbit of size 1, which contains all n triples with the isosceles difference pattern $(x, x, n - 2x)$. Similarly, a binary cyclic ordering is in an orbit of size 2, and those two Hamiltonian cycles contain all $2n$ triples from the orbits corresponding to that conjugate pair of difference patterns.

Lemma 11. *A unary ordering with difference pattern $(x, x, n - 2x)$ exists if and only if $\gcd(n, x) = 1$.*

Proof. A unary ordering with difference pattern $(x, x, n - 2x)$ is necessarily of the form

$$0 \quad x \quad 2x \quad \cdots \quad (n-1)x$$

(modulo n). Hence we require these n scalar multiples of x to all be distinct, i.e. that x is a generator for the additive group \mathbb{Z}_n . This happens if and only if n and x are coprime. \square

We remark that because $(x, x, n - 2x)$ and $(n - x, n - x, 2x)$ belong to the same equivalence class of orderings, they denote the same difference pattern. Thus the

number of isosceles difference patterns $(x, x, n - 2x)$ with $\gcd(n, x) = 1$ is $\frac{1}{2}\phi(n)$ (where ϕ denotes Euler's totient function).

Having characterised the unary orderings, we also have the following construction for binary orderings.

Lemma 12. *Suppose n is even, $x, y \in \mathbb{Z}_n$ are both odd and that $\gcd(x + y, n) = 2$. Then*

$$0 \quad x \quad x + y \quad 2x + y \quad 2x + 2y \quad \cdots \quad nx + (n - 1)y$$

is a binary ordering of \mathbb{Z}_n .

Proof. Since $\gcd(x + y, n) = 2$, the subgroup $H \leq \mathbb{Z}_n$ generated by $x + y$ has order $\frac{n}{2}$. Thus H contains all the even numbers in \mathbb{Z}_n . Since x is odd, the coset $H + x$ must contain all the odd numbers in \mathbb{Z}_n , and thus the list given is indeed an ordering of \mathbb{Z}_n .

To see that this ordering is binary, we observe that the differences between successive elements are alternately x and y . Thus the only difference patterns of triples that can appear are $(x, y, n - x - y)$ and $(y, x, n - x - y)$, so the ordering must be binary. \square

Lemma 13. *Let x, y and n be as in Lemma 12 above, and let C be the ordering of \mathbb{Z}_n constructed there. Then C and its translate $C + 1$ between them contain all triples with the conjugate pair of difference patterns $(x, y, n - x - y)$ and $(y, x, n - x - y)$.*

Proof. We observe that there are $n/2$ occurrences of each of the two specified difference patterns in each of C and $C + 1$. Also, no triple with one of those two difference patterns can appear in both orderings. Thus the set of triples in C with difference pattern $(x, y, n - x - y)$ is precisely

$$\{\{a, a + x, a + x + y\} \mid a \in \mathbb{Z}_n \text{ is even}\},$$

while the set of triples in $C + 1$ with difference pattern $(x, y, n - x - y)$ is precisely

$$\{\{b, b + x, b + x + y\} \mid b \in \mathbb{Z}_n \text{ is odd}\}.$$

A similar argument works for the conjugate difference pattern $(y, x, n - x - y)$. \square

So we are now almost ready to describe our difference pattern-based approach for finding Hamiltonian decompositions of $K_n^{(3)}$. However, we need one more definition before we continue:

Definition 14. A set of multifarious orderings $\{C_1, \dots, C_r\}$ (for some r) is called *compatible* if, for each $i \neq j$, $\tau(C_i) \cap \tau(C_j) = \emptyset$.

Consequently, the total number of difference patterns accounted for by a compatible set of size r is rn .

Now, our strategy for finding a Hamiltonian decomposition is as follows:

- Step 1. Find as large a set of r compatible multifarious orderings as possible, then take all n translates of each of these, to obtain rn Hamiltonian cycles which contain rn^2 distinct triples.
- Step 2. Examine the difference patterns that are “left over” (i.e. those which do not appear in the difference types of the compatible set), and if possible use the constructions for Hamiltonian cycles in Lemmas 11 and 12 to account for these.

This then gives us a more efficient method than clique-finding to carry out computer searches; again, we use GAP [11] to perform these. Consider the following examples.

Example 15. Let $n = 10$. Then $\binom{10}{3}/10 = 12$, so we would need only one multifarious ordering (and thus the issue of compatibility does not arise), and then need to account for two “left over” difference patterns. If these are both isosceles, or (since 10 is even) form a conjugate pair (in each case, satisfying the appropriate conditions), then we can apply either Lemma 11 or Lemma 12.

Using GAP to enumerate one example of each possible difference type of multifarious ordering, we find there are 36 cases to check. Four of these leave suitable conjugate pairs left over, while only one leaves two isosceles difference patterns left over.

One such example which leaves a conjugate pair left over is

0 1 8 4 2 3 6 7 9 5

so the 10 translates of this gives us 10 of the 12 Hamiltonian cycles we need. The leftover difference patterns are $(3,5,2)$ and $(5,3,2)$; using Lemma 12 we obtain the orderings

0 3 8 1 6 9 4 7 2 5 and
1 4 9 2 7 0 5 8 3 6

One can then verify that these 12 orderings (i.e. Hamiltonian cycles) do indeed give a Hamiltonian decomposition of $K_{10}^{(3)}$.

Alternatively, the ordering

0 1 3 5 6 9 4 8 2 7

leaves the isosceles difference patterns $(1, 1, 8)$ and $(3, 3, 4)$ leftover, so the 10 translates of this, together with

$$\begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & & \text{and} \\ 0 & 3 & 6 & 9 & 2 & 5 & 8 & 1 & 4 & 7 & & \end{array}$$

yield a different Hamiltonian decomposition of $K_{10}^{(3)}$.

The cases $n = 11$ and $n = 16$ can be handled similarly. When $n = 11$, we have $\binom{11}{3}/11 = 15$, so again we only need one multifarious ordering, and will have four “leftovers”.

Example 16. Let $n = 11$. Enumerating the possibilities of multifarious orderings of \mathbb{Z}_{11} shows that there 861 possible difference types. Of these, five leave only isosceles difference patterns left over. One such ordering is

$$0 \ 1 \ 5 \ 7 \ 2 \ 8 \ 4 \ 9 \ 6 \ 10 \ 3$$

which leaves the difference patterns $(1, 1, 9)$, $(2, 2, 7)$, $(3, 3, 5)$ and $(4, 4, 3)$ left over, which can easily be dealt with by using unary orderings.

When $n = 16$, we have $\binom{16}{3}/16 = 35$, so this time we will need a set of two compatible multifarious orderings, which will leave three difference patterns left over.

Example 17. Let $n = 16$. We find that the following set of compatible multifarious orderings

$$\begin{array}{cccccccccccccccc} 0 & 1 & 13 & 3 & 10 & 15 & 2 & 4 & 6 & 9 & 14 & 5 & 11 & 7 & 8 & 12 & \text{and} \\ 0 & 1 & 9 & 15 & 8 & 10 & 7 & 14 & 5 & 2 & 4 & 13 & 3 & 11 & 12 & 6 & \end{array}$$

leaves $(1, 1, 14)$, $(3, 3, 10)$ and $(5, 5, 6)$ left over. Again, these can be dealt with using unary orderings.

However, this method is not without its limitations. There are cases where Step 2 is not possible: the number of “left over” difference patterns can be more than the number that can be handled by our lemmas above. The first such case is $n = 13$: since $\binom{13}{3}/13 = 22$, we would need to account for $22 - 13 = 9$ leftover difference patterns. As 13 is odd, we can only use unary orderings, but there are only $\phi(13)/2 = 6$ possibilities for isosceles difference patterns. The case $n = 14$ also fails, as in each case there are not enough conjugate pairs available to us. Modifying our method to use $\mathbb{Z}_{n-1} \cup \{\infty\}$ or $\mathbb{Z}_{n-2} \cup \{\infty_1, \infty_2\}$ could potentially solve these cases, and we are investigating this.

6 Conclusion

The ideas in this paper were presented at the Ontario Combinatorics Workshop in May 2008, and subsequently at the Combinatorics 2008 conference in June 2008. At the latter meeting, A. Rosa communicated to the authors that he and M. Meszka had obtained, by computer search, Hamiltonian decompositions of $K_n^{(3)}$ for all feasible values of $n \leq 32$. Their results appear in [19].

We conclude the paper by summarising which sets of parameters (n, k) we have obtained Hamiltonian decompositions for $K_n^{(k)}$ in the table below:

n/k	2	3	4	5	6	7	8	9	10	11	12	13	14
5	W	W											
6	-	-	-										
7	W	C	C	W									
8	-	C	-	C	-								
9	W	-	C	C	-	W							
10	-	D	?	-	?	D	-						
11	W	D	?	?	?	?	D	W					
12	-	-	-	?	?	?	-	-	-				
13	W	MR	?	?	?	?	?	?	MR	W			
14	-	MR	-	?	-	-	-	?	-	MR	-		
15	W	-	?	-	-	?	?	-	-	?	-	W	
16	-	D	-	?	-	?	-	?	-	?	-	D	-

In the table above, the letters W, C, D and MR denote the method by which a Hamiltonian decomposition was found: W denotes Walecki's construction, C denotes the "clique-finding" method in Section 3, and D denotes the "difference pattern" method in Section 5, while MR denotes examples found by Meszka and Rosa [19]. A dash denotes an infeasible parameter set, while a question mark denotes that the parameters are feasible but no decomposition is known.

For all $n \geq 16$, apart from those examples (mentioned above) due to Meszka and Rosa, the problem of finding a Hamiltonian decomposition of $K_n^{(k)}$ (except for $k = 2$ and $k = n - 2$) remains open.

Appendix A: Further examples

Example 18. An example of a Hamiltonian decomposition of $K_8^{(3)}$:

1 2 5 3 8 7 4 6
1 3 5 7 8 6 4 2
1 4 5 2 8 3 6 7
1 5 7 4 3 2 6 8
1 6 5 8 4 2 7 3
1 7 2 5 6 4 3 8
1 8 2 7 6 5 3 4

Example 19. An example of a Hamiltonian decomposition of $K_9^{(4)}$:

1 2 3 7 9 8 4 5 6
1 2 5 4 8 7 6 3 9
1 2 7 8 3 4 5 6 9
1 3 5 7 9 8 6 4 2
1 4 5 7 6 2 3 9 8
1 4 7 8 2 6 3 5 9
1 6 3 4 9 8 2 7 5
1 6 3 7 4 9 2 5 8
1 6 7 4 9 5 3 8 2
1 7 8 5 6 2 4 3 9
1 7 9 5 2 3 4 8 6
1 8 2 9 6 7 5 3 4
1 8 5 9 6 2 4 7 3
1 9 6 8 3 5 7 2 4

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